Long-time regime and scaling of correlations in a simple model with glassy behaviour

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Abstract. We investigate the long-time regime of a zero-temperature kinetic model with glassy behaviour, introduced recently. In this regime, the energy density and its two-time correlation function are found to have simple asymptotic expressions with logarithmic corrections, which we derive exactly. In particular, the energy correlation function exhibits ageing. Its scaling form reads $C(t, t_w) \approx \sqrt{t_w/t}$, with a large correction in $1/\ln t_w$. We also find that a closely related model, with faster relaxation of the energy, leads to the same asymptotic scaling form for the two-time correlation, with much smaller corrections.

1. Introduction

Recently, a simple model of slow relaxation without energy barriers, presenting a number of the expected features of a glass, has been considered [1]. This model provides an interesting contribution to the investigation of the minimal ingredients necessary for glassy behaviour [2]. Its dynamics has then been studied analytically in [3–5].

The model is defined as follows. Consider N distinguishable particles. At the initial time $t = 0$ the particles are distributed amongst M boxes. At each time step $1/N$, a particle and a box are chosen independently at random. The particle is moved to the box if the energy, defined as minus the number of empty boxes, does not increase. In particular, moves from a box containing one particle to an empty box are allowed.

This is the zero-temperature version of the model introduced in [1]. It is a mean-field model, defined without any reference to a spatial structure. As shown in [4], the present model can be mapped onto a biased random-walk problem, thus permitting its analytical study.

In this paper we first analyse the dynamics of one-time quantities, thus completing the study begun in [4]. Analytic expressions are derived for the quantities of interest in the long-time regime, involving infinite series of logarithmic corrections. This study allows us, in particular, to investigate quantitatively the accuracy of the adiabatic approximation introduced in [4]. We then use this framework to find the scaling behaviour of the two-time energy correlation $C(t, t_w)$ studied in [1, 3, 5], starting from an expression derived in [5]. We finally compare these results to those found in a closely related model, corresponding to an unbiased random walk. We conclude by replacing the present study in the general context of off-equilibrium problems, such as the kinetics of phase-ordering or the dynamics of glasses.

2. Dynamical equations for one-time quantities

Let us denote by $n_k(t)$ the number of boxes containing k particles at time t. Conservation of the numbers of boxes and of particles yields

$$\sum_{k \geqslant 0} n_k(t) = M \tag{2.1a}$$

$$\sum_{k \geqslant 1} k\, n_k(t) = N\,. \tag{2.1b}$$

The fraction of boxes with k particles is $n_k(t)/M$. In the limit of an infinite system ($M \to \infty$), this quantity is self-averaging, i.e. its value for a single history becomes equal to its average over histories. We denote it by $f_k(t)$. From now on time is considered as continuous. We take as the initial condition the configuration where each box contains one particle, hence $M = N$ and

$$f_k(0) = \delta_{k,1}\,. \tag{2.2}$$

Equations (2.1) yield

$$\sum_{k \geqslant 0} f_k(t) = 1 \tag{2.3a}$$

$$\sum_{k \geqslant 1} k\, f_k(t) = 1\,. \tag{2.3b}$$

We denote by $\lambda(t)$ the mean size of non-empty boxes, i.e. their mean number of particles. We have

$$\frac{1}{\lambda(t)} = 1 - f_0(t) \tag{2.4}$$

while the energy of the system per box reads $\langle E(t) \rangle = -f_0(t)$ (see equation (5.4a)).

The time evolution of the probabilities $f_k(t)$ is given by [4]

$$\frac{\mathrm{d}}{\mathrm{d}t} f_k(t) = \frac{k+1}{\lambda(t)} f_{k+1}(t) + f_{k-1}(t) - \left(1 + \frac{k}{\lambda(t)} \right) f_k(t) \tag{2.5a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} f_1(t) = \frac{2}{\lambda(t)} f_2(t) - \left(1 + \frac{1}{\lambda(t)} \right) f_1(t) \tag{2.5b}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} f_0(t) = \frac{1}{\lambda(t)} f_1(t)\,. \tag{2.5c}$$

These equations express the balance between gains and losses to the probabilities $f_k(t)$. Equations (2.4) and (2.5c) imply

$$\frac{\mathrm{d}}{\mathrm{d}t} \lambda(t) = \lambda(t) f_1(t)\,. \tag{2.6}$$

Equations (2.5) describe an asymmetric random walk on the half-line with an absorbing site at the origin, the size k of boxes playing the role of the discrete position of the random walker. Indeed they can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t} f_k(t) = \mu_{k+1} f_{k+1}(t) + \lambda_{k-1} f_{k-1}(t) - (\mu_k + \lambda_k) f_k(t) \tag{2.7}$$

where $\mu_k = k/\lambda(t)$ and $\lambda_k = 1$ are the transition rates from site k ($k \neq 0$), to the left and to the right, respectively, and $\lambda_0 = \mu_0 = 0$. The velocity and diffusion coefficient of this random walk read

$$V_k = \lambda_k - \mu_k = 1 - k/\lambda(t) \qquad D_k = (\lambda_k + \mu_k)/2 = [1 + k/\lambda(t)]/2. \tag{2.8}$$

Hence, according to whether the size k of the box is smaller or larger than the mean size $\lambda(t)$, the bias $b_k = V_k/(2D_k) = (\lambda(t) - k)/(\lambda(t) + k)$ is positive or negative. This bias generates entropy barriers [4].

3. Generating-function formalism

We consider the generating function

$$G(x, t) = \sum_{k \geqslant 0} f_k(t) x^k. \tag{3.1}$$

It obeys the equation

$$\frac{\partial}{\partial t} G(x, t) = (x - 1) \left(G(x, t) - \frac{1}{\lambda(t)} \frac{\partial}{\partial x} G(x, t) - f_0(t) \right) \tag{3.2}$$

with the initial condition inherited from (2.2)

$$G(x, 0) = x. \tag{3.3}$$

We also have

$$G(0, t) = f_0(t) \qquad G(1, t) = 1. \tag{3.4}$$

Equation (3.2) can be solved, at least formally, by the method of characteristics. Define

$$y(x, t) = (1 - x) e^{-\tau(t)} \tag{3.5}$$

with

$$\tau(t) = \int_0^t \frac{du}{\lambda(u)}. \tag{3.6}$$

Then $F(y, t) = G(x, t)$ obeys

$$\frac{\partial}{\partial t} F(y, t) = -y e^{\tau(t)} [F(y, t) - f_0(t)]. \tag{3.7}$$

Defining

$$D(t, s) = \int_s^t e^{\tau(u) - \tau(t)} du \tag{3.8}$$

one finds

$$G(x, t) = e^{(x-1)D(t,0)} \left(1 + (x-1)e^{-\tau(t)}\right) + (1-x)\int_0^t du\, f_0(u)\, e^{\tau(u)-\tau(t)}\, e^{(x-1)D(t,u)}.$$

$$(3.9)$$

Equation (3.9) can be rewritten as

$$G(x, t) = e^{(x-1)D(t,0)} \left(1 + (x-1)e^{-\tau(t)}\right) + \int_0^t du\, f_0(u)\frac{d}{du}\, e^{(x-1)D(t,u)}$$

$$= e^{(x-1)D(t,0)} \left(1 + (x-1)e^{-\tau(t)}\right) + f_0(t) - \int_0^t du\, \frac{f_1(u)}{\lambda(u)}\, e^{(x-1)D(t,u)}.$$ $$(3.10)$$

Hence, using the first of equations (3.4), one obtains an implicit non-linear integral equation for $\lambda(t)$

$$1 - e^{-\tau(t)} = \int_0^t du\, \frac{f_1(u)}{\lambda(u)}\, e^{D(t,0)-D(t,u)}$$ $$(3.11)$$

with

$$D(t, 0) - D(t, u) = \int_0^u e^{\tau(v)-\tau(t)}\, dv.$$ $$(3.12)$$

The above generating-function formalism was already present in [4, 5]. It has been used in [5] to derive (3.9) from (2.5).

4. Long-time regime for one-time quantities

The aim of this section is to obtain an analytical description of the long-time behaviour of $\lambda(t)$ and of the probabilities $f_k(t)$. This can be done by three different methods, as follows.

(i) The first approach is based on an analysis of the formal solution (3.11) in the long-time regime. It only uses the fact that quantities such as $\lambda(t)$ are slowly varying in time. The result (4.7) will provide a justification of this assertion.

Neglecting the exponentially small term in the left-hand side of (3.11), we obtain

$$1 = \int_0^t du\, \frac{f_1(u)}{\lambda(u)} + \int_0^t du\, \frac{f_1(u)}{\lambda(u)} \left(e^{D(t,0)-D(t,u)} - 1\right).$$ $$(4.1)$$
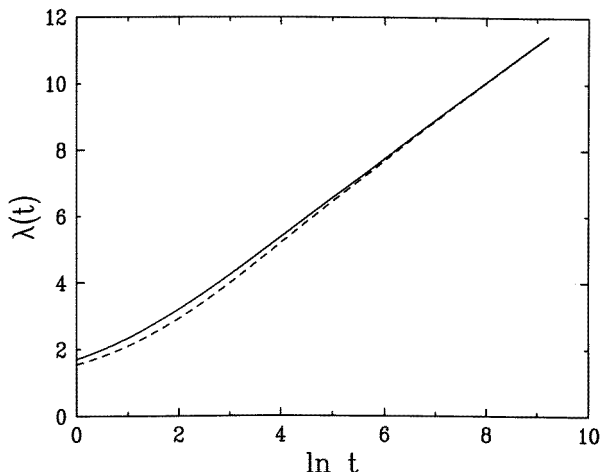
The first integral in the right-hand side is equal to $1 - 1/\lambda(t)$, by equation (2.6). The second integral is dominated by values of u close to the upper bound t. We therefore set $u = t - \varepsilon$, with $\varepsilon \ll t$, and we use the following estimates:

$$\tau(t) - \tau(u) = \int_u^t \frac{dv}{\lambda(v)} \approx \frac{\varepsilon}{\lambda(t)}$$ $$(4.2a)$$

$$D(t, u) = \int_u^t e^{\tau(v)-\tau(t)}\, dv \approx \lambda(t)\left(1 - e^{-\varepsilon/\lambda(t)}\right).$$ $$(4.2b)$$

Setting $z = \exp[-\varepsilon/\lambda(t)]$, and again using (2.6), we obtain

$$\frac{dt}{d\lambda} = \frac{1}{\lambda(t)f_1(t)} \approx I(\lambda(t))$$ $$(4.3)$$

Figure 1. Plot of the mean size $\lambda(t)$ of the non-empty boxes at time t, against $\ln t$. Full curve: finite-time data, obtained by numerical integration of the evolution equations (2.5), up to $t = 10\,000$. Broken curve: analytical prediction (4.5) of the long-time analysis.

where

$$I(\lambda) = \int_0^1 \frac{\mathrm{d}z}{z}(\mathrm{e}^{\lambda z} - 1) = \sum_{n \geqslant 1} \frac{\lambda^n}{n\,n!} = \mathrm{Ei}(\lambda) - \ln \lambda - \gamma \sim \frac{\mathrm{e}^\lambda}{\lambda} \sum_{\ell \geqslant 0} \frac{\ell!}{\lambda^\ell}. \tag{4.4}$$

In this equation $\mathrm{Ei}(x)$ denotes the exponential integral, and γ is Euler's constant. The last expression is an asymptotic expansion, valid for λ large.

Equation (4.3) yields the relation between t and λ as

$$t(\lambda) - t_0 \approx \int_0^\lambda I(\lambda')\,\mathrm{d}\lambda' = \int_0^1 \frac{\mathrm{d}z}{z^2}(\mathrm{e}^{\lambda z} - 1 - \lambda z) = \sum_{n \geqslant 1} \frac{\lambda^{n+1}}{n\,(n+1)!} \sim \frac{\mathrm{e}^\lambda}{\lambda} \sum_{\ell \geqslant 0} \frac{(\ell+1)!}{\lambda^\ell}. \tag{4.5}$$

A natural choice for the integration constant t_0 consists in fixing $t(1) = 0$.

The long-time expression (4.5) is the main prediction of this section. The relative error made in the above analysis is roughly of order $\mathrm{d}\lambda/\mathrm{d}t$. This quantity itself is of order $\mathrm{e}^{-\lambda}$ by (4.3) and (4.4), or of order $1/t$. We thus expect our results (4.4), (4.5) to be valid up to a relative error exponentially small in λ. As a consequence, the last expressions in the right-hand sides of (4.4), (4.5) yield the correct asymptotic power-series expansions to all orders in $1/\lambda$. Keeping only the first two terms, we have

$$t = \frac{\mathrm{e}^\lambda}{\lambda}\left[1 + \frac{1}{\lambda} + \mathcal{O}\left(\frac{1}{\lambda^2}\right)\right] \tag{4.6}$$

i.e.

$$\lambda(t) = \ln t + \ln \ln t + \cdots \tag{4.7}$$

where the higher-order terms roughly vanish as $1/\ln t$ when $t \to \infty$.

In figure 1 we give a comparison of the long-time prediction (4.5) with finite-time data for $\lambda(t)$, obtained by a numerical integration of the differential equations (2.5), up to $t = 10\,000$. This graph was already given in [4]. It is presented here for further comparison with figure 3. It shows that the long-time analysis exposed above yields a good description of the behaviour of $\lambda(t)$, even for times as small as $t \sim 1$.

(ii) Another approach, based on an adiabatic approximation, has been presented in [4]. It consists in considering λ as a constant in order to compute the mean absorption time of the random-walk problem (2.7). This method yields (4.5). However, no analysis of the accuracy of the adiabatic approximation was given in [4]. The above approach provides such an analysis. A prior adiabatic approximation to the model studied here was presented in [3]. It leads to a result similar to equation (4.10) below.

(iii) We find it interesting to present a third derivation of equation (4.5). This approach again uses the fact that λ is slowly varying in time.

As a first approximation, we look for a stationary solution to equations (2.5a) and (2.5b), with no absorption at $k = 0$, obtaining $f_k = C\lambda^k/k!$. The normalization condition (2.3b) then fixes the constant C. We thus obtain the Poisson distribution

$$f_k \approx f_k{}^{\text{Poisson}} = \mathrm{e}^{-\lambda}\frac{\lambda^{k-1}}{k!} \qquad (k \geqslant 1). \tag{4.8}$$

The only time dependence of the probabilities $f_k{}^{\text{Poisson}}$ is contained in the dependence of λ on t. These probabilities are peaked around $k = \lambda$, with a Gaussian scaling form of width $\sqrt{\lambda}$, namely

$$f_k{}^{\text{Poisson}} \approx \frac{1}{\sqrt{2\pi\lambda^3}} \exp\left(-\frac{(k-\lambda)^2}{2\lambda}\right). \tag{4.9}$$

As a consequence, the evolution of $\lambda(t)$, given by equation (2.6), is dominated by large deviations, i.e. very rare events in the long-time regime [4]. Within the present approximation we have $\mathrm{d}\lambda/\mathrm{d}t \approx \lambda f_1{}^{\text{Poisson}} = \lambda\,\mathrm{e}^{-\lambda}$, hence

$$\frac{\mathrm{d}t}{\mathrm{d}\lambda} \approx \frac{\mathrm{e}^{\lambda}}{\lambda} \tag{4.10}$$

in agreement with equations (4.3), (4.4), to leading order.

The above derivation, already given in [4], can be improved systematically by looking for a stationary solution to equations (2.5a) and (2.5b) which is more accurate than the $f_k{}^{\text{Poisson}}$. To do so, we solve these equations iteratively as follows. Assuming that f_1 is known, we obtain $f_2 = (\lambda + 1)f_1/2$, and so on, i.e.

$$f_k = f_1\frac{\lambda^{k-1}}{k!}\sum_{\ell=0}^{k-1}\frac{\ell!}{\lambda^\ell}. \tag{4.11}$$

The above solution is more accurate than the Poisson distribution (4.8), as long as k is not too large. Indeed for $k \ll \lambda$ (more precisely $\lambda - k \gg \sqrt{\lambda}$) the sum in the right-hand side of (4.11) is a small correction to its prefactor, while for $k \gg \lambda$ (more precisely $k - \lambda \gg \sqrt{\lambda}$) equation (4.11) yields the unacceptable behaviour $f_k \approx f_1/k$. We thus choose to match the improved solution (4.11) with the Poisson distribution (4.8) at some point $k = K \ll \lambda$. This yields

$$\frac{\mathrm{d}t}{\mathrm{d}\lambda} \approx \frac{\mathrm{e}^{\lambda}}{\lambda}\sum_{\ell=0}^{K-1}\frac{\ell!}{\lambda^\ell}. \tag{4.12}$$

This expression correctly predicts the first K correction terms in $1/\lambda$ of the long-time solution (4.4). If we include more and more corrections by letting K go to infinity, we recover the (4.3), with $I(\lambda)$ given by the series $I(\lambda) = \mathrm{e}^{\lambda}S(\lambda)$, where

$$S(\lambda) = \sum_{\ell \geqslant 0}\frac{\ell!}{\lambda^{\ell+1}}. \tag{4.13}$$

To summarize the three approaches described above, we note that (i) and (ii) provide the closed-form long-time solution (4.5), while (iii) yields the series (4.13). This series is asymptotic but divergent. It is actually one of the first historical examples of a divergent asymptotic series [6], already considered by Euler. The sum $S(\lambda)$ is only defined up to an exponentially small ambiguity S_{amb}, of order the smallest of the terms $\ell!/\lambda^{\ell+1}$. The minimum takes place for $\ell \approx \lambda$, and therefore $S_{\mathrm{amb}} \approx \mathrm{e}^{-\lambda}$. The ambiguity of the sum $S(\lambda)$ is thus of relative order $\mathrm{e}^{-\lambda} \sim 1/t$, which is precisely the order of magnitude of the error to the prediction (4.5) of the long-time analysis.

Let us note that, in a more general context, the presence of exponentially small corrections, which are non-perturbative in the control parameter of the problem, here $\lambda(t)$, is often met when using adiabatic approximations. This is, for instance, the case in classical mechanics.

5. Dynamical equations and long-time regime for correlations

We now consider correlations between configurations of the system at two different times t_{w} and t, with $0 \leqslant t_{\mathrm{w}} \leqslant t$. Along the lines of [5], we denote by $g_k(t, t_{\mathrm{w}})$ the probability that any given box contains k particles at time t, knowing that it was empty at the earlier time t_{w}. We have

$$\sum_{k \geqslant 0} g_k(t, t_{\mathrm{w}}) = 1 \tag{5.1}$$

and

$$g_k(t_{\mathrm{w}}, t_{\mathrm{w}}) = \delta_{k,0}. \tag{5.2}$$

Let $k_i(t)$ be the number of particles contained in box i at time t. The energy per box at time t then reads

$$E(t) = -\frac{1}{M} \sum_{i=1}^{M} \delta_{k_i(t),0} \tag{5.3}$$

since the sum counts the number of empty boxes at time t. As long as the number M of boxes is finite, the energy per box $E(t)$ is a fluctuating quantity depending on the history of the system. Its first moments and its correlation function can be expressed as

$$\langle E(t) \rangle = -f_0(t) \tag{5.4a}$$

$$\langle E(t)^2 \rangle = f_0(t)^2 + \frac{1}{M} f_0(t)[1 - f_0(t)] \tag{5.4b}$$

$$\langle E(t)E(t_{\mathrm{w}}) \rangle = f_0(t)\, f_0(t_{\mathrm{w}}) + \frac{1}{M} f_0(t_{\mathrm{w}})[g_0(t, t_{\mathrm{w}}) - f_0(t)]. \tag{5.4c}$$

Hence the connected two-time correlation function of the model, defined as [1, 3, 5]

$$C(t, t_{\mathrm{w}}) = \frac{\langle E(t)E(t_{\mathrm{w}}) \rangle - \langle E(t) \rangle \langle E(t_{\mathrm{w}}) \rangle}{\langle E(t_{\mathrm{w}})^2 \rangle - \langle E(t_{\mathrm{w}}) \rangle^2} \tag{5.5}$$

reads

$$C(t, t_{\mathrm{w}}) = \frac{g_0(t, t_{\mathrm{w}}) - f_0(t)}{1 - f_0(t_{\mathrm{w}})} = \lambda(t_{\mathrm{w}})[g_0(t, t_{\mathrm{w}}) - f_0(t)]. \tag{5.6}$$

The initial value (5.2) implies $C(t_w, t_w) = 1$.

As shown in [5], differential evolution equations for the g_k can be derived by conditioning the balance equations (2.5) for the f_k on the boxes which are empty at time t_w. One thus obtains

$$\frac{d}{dt} g_k(t, t_w) = \frac{k+1}{\lambda(t)} g_{k+1}(t, t_w) + g_{k-1}(t, t_w) - \left(1 + \frac{k}{\lambda(t)}\right) g_k(t, t_w) \tag{5.7a}$$

$$\frac{d}{dt} g_1(t, t_w) = \frac{2}{\lambda(t)} g_2(t, t_w) + f_1(t) g_0(t, t_w) - 2g_1(t, t_w) \tag{5.7b}$$

$$\frac{d}{dt} g_0(t, t_w) = g_1(t, t_w) - f_1(t) g_0(t, t_w). \tag{5.7c}$$

These are the same equations as equations (2.5), i.e. they describe a random walk with the same transition rates, except for $k = 0$ and $k = 1$, where now $\lambda_0 = f_1$, $\mu_1 = 1$, and $\mu_0 = 0$. Without conditioning, equations (5.7) reduce to (2.5), and the g_k are identical to the f_k. Let us note that the time t_w, entering the initial condition (5.2), plays the role of a parameter in these equations.

The generating function

$$H(x, t, t_w) = \sum_{k \geqslant 0} g_k(t, t_w) x^k \tag{5.8}$$

obeys the equation

$$\frac{\partial}{\partial t} H(x, t, t_w) = (x - 1) \left(H(x, t, t_w) - \frac{1}{\lambda(t)} \frac{\partial}{\partial x} H(x, t, t_w) - h(t, t_w) \right) \tag{5.9}$$

where

$$h(t, t_w) = g_0(t, t_w)[1 - f_1(t)] + g_1(t, t_w) f_0(t) \tag{5.10}$$

and with the initial condition

$$H(x, t_w, t_w) = 1. \tag{5.11}$$

We then have

$$H(0, t, t_w) = g_0(t, t_w) \qquad H(1, t, t_w) = 1. \tag{5.12}$$

Using again the method of characteristics [5], we can derive the following formal solution to (5.9):

$$H(x, t, t_w) = e^{(x-1)D(t,t_w)} + (1 - x) \int_{t_w}^t du\, h(u, t_w)\, e^{\tau(u) - \tau(t)}\, e^{(x-1)D(t,u)}. \tag{5.13}$$

The boundary condition (5.12) at $x = 0$ leads to the implicit integral equation

$$g_0(t, t_w) = 1 + \int_{t_w}^t du\, (h(u, t_w) - 1)\, e^{\tau(u) - \tau(t)}\, e^{-D(t,u)}$$

$$= 1 + \int_{t_w}^t du\, (h(u, t_w) - 1)\, \frac{d}{du}\, e^{-D(t,u)}. \tag{5.14}$$

We now turn to the analysis of the energy correlation function $C(t, t_w)$ in the asymptotic regime where t and t_w go to infinity. Our starting point is the exact relation (5.14), and its analysis closely follows that of (3.11), carried out in section 4. We again set $u = t - \varepsilon$, and we perform the Taylor expansion

$$h(u, t_w) = h(t, t_w) - \varepsilon \frac{d}{dt} h(t, t_w) + \cdots. \tag{5.15}$$

The choice of keeping two terms will become clear after equation (5.22). By inserting this expansion into equation (5.14), we obtain

$$g_0(t, t_w) \approx 1 + (h(t, t_w) - 1) J_0 - \frac{d}{dt} h(t, t_w) J_1 + \cdots. \tag{5.16}$$

The first integral equals $J_0 = 1 - \exp(-D(t, t_w)) \approx 1 - e^{-\lambda}$. The second integral can be estimated by means of (4.2), hence

$$J_1 \approx \int_0^\infty d\varepsilon \, \varepsilon \, e^{\varepsilon/\lambda} \, \exp[\lambda (1 - e^{-\varepsilon/\lambda})] = \lambda \, e^{-\lambda} \, I(\lambda). \tag{5.17}$$

Thus

$$g_0(t, t_w) \approx e^{-\lambda} + h(t, t_w) \left(1 - e^{-\lambda}\right) - \lambda \, e^{-\lambda} \, I(\lambda) \frac{d}{dt} h(t, t_w). \tag{5.18}$$

This estimate remains to be combined with the relations (5.7c) and (5.10). Setting

$$N_0(t, t_w) = g_0(t, t_w) - f_0(t) = \frac{C(t, t_w)}{\lambda(t_w)} \tag{5.19}$$

and neglecting all terms which are exponentially small in λ with respect to the leading ones, we finally get

$$\frac{d}{dt} N_0(t, t_w) \approx - \frac{\lambda \, e^{-\lambda} + f_1(t)}{1 - \lambda + \lambda^2 \, e^{-\lambda} \, I(\lambda)} \, N_0(t, t_w) \tag{5.20}$$

or equivalently, in terms of $C(t, t_w)$ and changing variables from t to λ

$$\frac{d}{d\lambda} C \approx -\alpha(\lambda) \, C \tag{5.21}$$

with

$$\alpha(\lambda) = \frac{\lambda \, e^{-\lambda} + f_1}{\lambda \, f_1 \left[1 - \lambda + \lambda^2 \, e^{-\lambda} \, I(\lambda)\right]} = \frac{1}{\lambda} + \frac{e^\lambda}{(1 - \lambda) e^\lambda + \lambda^2 I(\lambda)}. \tag{5.22}$$

We can now justify *a posteriori* why we have kept the first two terms in the expansion (5.15). Indeed it can be checked that the contributions to $\alpha(\lambda)$ of higher-order derivatives of $h(t, t_w)$ with respect to t are exponentially negligible.

Finally, we find that the energy correlation function obeys the scaling law

$$C(t, t_w) \approx \frac{\varphi(t_w)}{\varphi(t)} = \frac{\Phi[\lambda(t_w)]}{\Phi[\lambda(t)]} \tag{5.23}$$

with

$$\varphi(t) = \Phi[\lambda(t)] = \exp\left(\int_0^{\lambda(t)} \alpha(\lambda')\, d\lambda'\right). \tag{5.24}$$

The expansion

$$\alpha(\lambda) = \frac{1}{2}\left[1 + \frac{1}{\lambda} + \mathcal{O}\left(\frac{1}{\lambda^2}\right)\right] \tag{5.25}$$

yields

$$\varphi(t) = \Phi[\lambda(t)] = \sqrt{\lambda(t)\, e^{\lambda(t)}}\left[1 + \mathcal{O}\left(\frac{1}{\lambda(t)}\right)\right] = \sqrt{t}\,\ln t\left[1 + \mathcal{O}\left(\frac{1}{\ln t}\right)\right] \tag{5.26}$$

hence

$$C(t, t_w) = \sqrt{\frac{t_w}{t}}\,\frac{\ln t_w}{\ln t}\left[1 + \mathcal{O}\left(\frac{1}{\ln t_w},\, \frac{1}{\ln t}\right)\right]. \tag{5.27}$$

In terms of the dimensionless ratio

$$x = \frac{t}{t_w} \geqslant 1 \tag{5.28}$$

the above result reads

$$C(t, t_w) = \frac{1}{\sqrt{x}}\left[1 - \frac{\ln x}{\ln t_w} + \mathcal{O}\left(\frac{1}{(\ln t_w)^2}\right)\right]. \tag{5.29}$$

The above expressions for the correlation function are the main results of this section. Let us emphasize that the full scaling prediction (5.23) of the long-time analysis incorporates all orders of an asymptotic series expansion in $1/\lambda$, just as (4.5).

Figure 2 shows logarithmic plots of the correlation function $C(t, t_w)$ against the ratio x, for several values of t_w. The finite-time data were obtained by a numerical integration of the differential equations (5.7). In agreement with equation (5.29), they exhibit a very slow logarithmic convergence toward the exact limit $1/\sqrt{x}$.
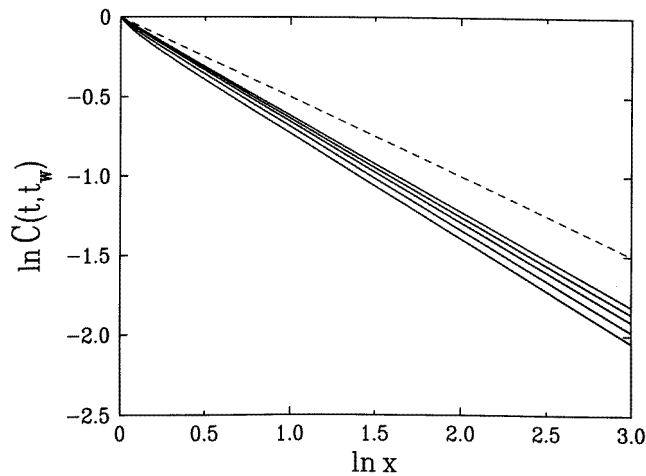
Figure 2. Log–log plot of the energy correlation function $C(t, t_w)$, against the ratio $x = t/t_w$. Full curves: finite-time data, obtained by numerical integration of the evolution equations (2.5) and (5.7), for $t_w = 10, 30, 100, 300$, and 1000 (from bottom to top). Broken line with slope $-\frac{1}{2}$: exact $t_w \to \infty$ limit value $C(t, t_w) = 1/\sqrt{x}$.
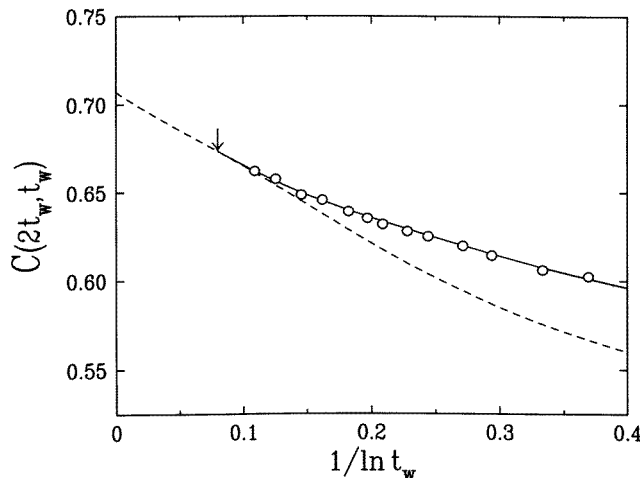
Figure 3. Plot of the energy correlation function $C(t, t_w)$ for $x = t/t_w = 2$, against $1/\ln t_w$. Full curve: finite-time data, obtained by numerical integration of the evolution equations (2.5) and (5.7), up to $t = 2t_w = 512\,000$ (arrow). Circles: Monte Carlo data (see text). The error bar on the data points is comparable to the size of the symbols. Broken curve: analytical prediction (5.23) of the long-time analysis.

A more precise study of this convergence is given in figure 3 for the value $x = 2$, already considered in [1]. Finite-time data for the correlation function $C(2t_w, t_w)$, up to $t = 2t_w = 512\,000$, are plotted against $1/\ln t_w$, together with the long-time result (5.23), in order to better show their convergence to the limit value $1/\sqrt{2}$. The finite-time data are fully confirmed by the results of a Monte Carlo simulation of the dynamics of the system, averaged over 100 histories, for a number of boxes and particles equal to $10\,000$, and a maximal time $t = 2t_w = 19\,200$. A comparison of figures 1 and 3 shows that full quantitative agreement between the finite-time data and the long-time result (5.23) only sets in at much longer times than for $\lambda(t)$.

Let us point out that the presence of logarithmic corrections to scaling makes any numerical analysis of the long-time regime very difficult. This difficulty is illustrated by figure 2 where the data exhibit, over a significant range of values of x, and for values of t_w between 10 and 1000, an apparent power-law fall-off with a larger negative exponent than the exact value $-\frac{1}{2}$. This presumably explains the scaling form proposed in [1] for the energy correlation for $t \gg t_w$, namely $C(t, t_w) \sim (t_w/t)^{0.70}$, instead of the exact square-root law (5.29). Note also the incorrect prediction 0.58 for the limit of $C(2t_w, t_w)$ as $t_w \to \infty$ found in this reference, instead of the exact value $1/\sqrt{2}$.

Remark. Let us show that equation (5.22) can be derived alternatively along the lines of approach (iii) of section 4. To do so, we generalize (5.19) by setting

$$N_k(t, t_w) = g_k(t, t_w) - f_k(t) . \tag{5.30}$$

As a first approximation, we look for a stationary solution to equation (5.7a), thus again obtaining a Poisson law,

$$N_k \approx \frac{A\lambda^k}{k!} \qquad (k \geqslant 2) . \tag{5.31}$$

Equation (5.7*b*), together with (5.1) and (2.3*a*), determine both N_1 and the constant A in terms of N_0. Equation (5.7*c*) then leads to the evolution equation

$$\frac{\mathrm{d}}{\mathrm{d}t} N_0 \approx -\frac{\lambda + 1}{2} \mathrm{e}^{-\lambda} N_0 \tag{5.32}$$

in agreement with both terms in the expansion (5.25).

   This derivation can again be improved by incorporating systematic corrections to all orders in $1/\lambda$. Indeed, assuming f_1 and $\rho_1 = N_1/N_0$ to be known, we can solve the stationary form of (5.7*a*) iteratively, thus obtaining

$$N_k = N_0 \frac{\lambda^k}{k!} \left(\frac{\rho_1}{\lambda} + (\rho_1 - f_1) \sum_{\ell=1}^{k-1} \frac{\ell!}{\lambda^\ell} \right). \tag{5.33}$$

The argument then goes on as in section 4. We match the expressions (5.31) and (5.33) at some point $k = K \ll \lambda$, using equations (5.1) and (2.3*a*). This yields the expression (5.22) for $\alpha(\lambda)$, but with $I(\lambda) = \mathrm{e}^\lambda S(\lambda)$ given in terms of the asymptotic series (4.13). The discussion presented at the end of section 4 therefore applies to the energy correlation function as well.

6. Discussion

Let us first compare the model studied so far, called model A in [4], with a closely related model, called model B there, and defined as follows. The particles are no longer distinguishable. At each time step a box is chosen at random among the non-empty boxes and one of its particles is moved to another randomly chosen box if the energy (defined as minus the number of empty boxes) does not increase. The probabilities f_k have been shown [4] to perform a symmetric (unbiased) random walk, described by the equations

$$\frac{\mathrm{d}}{\mathrm{d}t} f_k(t) = f_{k+1}(t) + f_{k-1}(t) - 2 f_k(t) \tag{6.1a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} f_1(t) = f_2(t) - 2 f_1(t) \tag{6.1b}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} f_0(t) = f_1(t). \tag{6.1c}$$

It can be shown analogously that the $g_k(t, t_\mathrm{w})$ obey [7]

$$\frac{\mathrm{d}}{\mathrm{d}t} g_k(t, t_\mathrm{w}) = g_{k+1}(t, t_\mathrm{w}) + g_{k-1}(t, t_\mathrm{w}) - 2 g_k(t, t_\mathrm{w}) \tag{6.2a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} g_1(t, t_\mathrm{w}) = g_2(t, t_\mathrm{w}) + \lambda(t) f_1(t) g_0(t, t_\mathrm{w}) - [1 + \lambda(t)] g_1(t, t_\mathrm{w}) \tag{6.2b}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} g_0(t, t_\mathrm{w}) = \lambda(t) g_1(t, t_\mathrm{w}) - \lambda(t) f_1(t) g_0(t, t_\mathrm{w}). \tag{6.2c}$$

   The long-time analysis of model B, the details of which will be reported elsewhere [7], leads to the following results. First, the relaxation of the energy is much faster. Indeed, we have

$$\lambda(t) = \sqrt{\pi t} \left(1 + \frac{1}{16t} + \cdots \right). \tag{6.3}$$

Moreover, the two-time energy correlation has the same asymptotic scaling form as that derived above for model A, though with much smaller corrections, namely

$$C(t, t_w) = \frac{1}{\sqrt{x}} \left[1 - \frac{\sqrt{x-1}}{x\sqrt{\pi}} \frac{1}{\sqrt{t_w}} + \mathcal{O}\left(\frac{1}{t_w}\right) \right] \tag{6.4}$$

where $x = t/t_w$.

The above results can be replaced in the general context of the study of off-equilibrium systems. Consider first the kinetics of phase ordering [8]. In coarsening systems, a scaling hypothesis, proved for simple models and well verified by a number of other ones, states that the long-time dynamics is characterized by a single time-dependent length scale $L(t)$, the typical size of domains. As a consequence, the distribution of domain sizes is independent of time, when lengths are scaled by $L(t)$, and the two-time autocorrelation function takes the form

$$C(t, t_w) \approx F\left(\frac{L(t)}{L(t_w)}\right). \tag{6.5}$$

One usually has [8, 9]

$$F(y) \sim y^{-\Lambda} \qquad \text{for} \quad y = L(t)/L(t_w) \gg 1. \tag{6.6}$$

Moreover, if $L(t) \sim t^\alpha$, then (6.5) yields

$$C(t, t_w) \approx G\left(\frac{t}{t_w}\right) \tag{6.7}$$

and, using (6.6)

$$G(x) \sim x^{-\alpha\Lambda} \qquad \text{for} \quad x = t/t_w \gg 1. \tag{6.8}$$

This scaling picture applies to model B. The mean size $\lambda(t)$ of the non-empty boxes plays the role of $L(t)$. The distribution of box sizes obeys a scaling law [4]. Finally, according to (6.3) and (6.4), one has $\alpha = \frac{1}{2}$ and $F(y) = 1/y$, hence $\Lambda = 1$ and $G(x) = 1/\sqrt{x}$.

The case of model A, studied in this work, is more complex. The bias in the random walk (2.5) is responsible for the existence of entropy barriers [4]. There is no time-independent distribution of box sizes k in the scaling variable k/λ (see equation (4.9)). As a consequence, the dynamics of this model is entirely driven by very rare events, so that the mean size $\lambda(t)$ only grows logarithmically. Furthermore, the correlation function does not obey (6.5). Instead, one has (see equation (5.23))

$$C(t, t_w) \approx \frac{\varphi(t_w)}{\varphi(t)} \tag{6.9}$$

with $\varphi(t) \approx \sqrt{t} \ln t$, which shows that the mean size $\lambda(t)$ is not the relevant length scale for correlations. Nevertheless, forgetting about logarithmic corrections, we have $\varphi(t) \sim \sqrt{t}$, so that (6.9) is of the power-law form (6.7), with $G(x) = 1/\sqrt{x}$. Correlations thus have a power-law behaviour in t/t_w, even though the mean size grows logarithmically.

Finally, we note that the two-time correlation functions of models A and B are given asymptotically by the same power law $C(t, t_w) \approx \sqrt{t_w/t}$, although their dynamics are very different.

Let us conclude by mentioning that the scaling forms (6.5) and (6.9) can be viewed as special cases of the more general law

$$C(t, t_w) \approx F\left(\frac{h(t)}{h(t_w)}\right) \tag{6.10}$$

which is met in several instances, such as the mode-coupling approach to relaxation in glasses [10], and the dynamics of the p-spin spherical model [11, 12]. The present work thus provides an example where the function $h(t)$ is known explicitly.

Acknowledgments

References

[1]   Ritort F 1995 *Phys. Rev. Lett.* **75** 1190
[2]   van Hemmen J L and Morgenstern I (ed) 1987 *Heidelberg Colloquium on Glassy Dynamics* (Berlin: Springer)
       Angell C A 1995 *Science* **267** 1924
[3]   Franz S and Ritort F 1995 *Europhys. Lett.* **31** 507
[4]   Godrèche C, Bouchaud J P and Mézard M 1995 *J. Phys. A: Math. Gen.* **28** L603
[5]   Franz S and Ritort F 1995 *Preprint* cond-mat 9508133
[6]   Hardy G H 1949 *Divergent Series* (Oxford: Clarendon)
[7]   Godrèche C and Luck J M in preparation
[8]   Bray A J 1994 *Adv. Phys.* **43** 357
[9]   Fisher D S and Huse D A 1988 *Phys. Rev.* B **38** 373
[10] Bouchaud J P, Cugliandolo L F, Kurchan J and Mézard M 1995 *Preprint* cond-mat 9511042
[11] Cugliandolo L F and Kurchan J 1993 *Phys. Rev. Lett.* **71** 173; 1994 *J. Phys. A: Math. Gen.* **27** 5749
[12] Franz S and Mézard M 1994 *Europhys. Lett.* **26** 209; 1994 *Physica* **210A** 48